

REMARK ON ALGORITHM 292 [S22]*
 REGULAR COULOMB WAVE FUNCTIONS [Walter
 Gautschi, *Comm. ACM* 9 (Nov. 1966), 793]
 AND ON

REMARK ON ALGORITHM 292 [S22]
 REGULAR COULOMB WAVE FUNCTIONS [Walter
 Gautschi, *Comm. ACM* 12 (May 1969), 280]

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 functions, regular Coulomb wave functions
 CR CATEGORIES: 5.12

The revised version of the procedure *Coulomb* was translated
 into IBM System/360 Algol and tested on an IBM S/360 Model 75
 Computer. When $\eta > 12$ overflow problems were encountered in
 the generation of intermediate arrays. These were due to the
 smaller exponent range of the S/360, $-64 \leq \text{exp} \leq 63$. The follow-
 ing changes, while not completely eliminating the overflow prob-
 lems, greatly alleviate them.

Insert **real scale**;
 after **begin integer** *L, nu, nu1, mu, mu1, i, k*;
 Insert *scale := 16 ↑ (-57)*;
comment This value of *scale* is appropriate for the IBM S/360.
 On a machine with a different base and a different exponent
 range, say $\alpha \leq \text{exp} \leq \beta$, the value of *scale* should be *base* ↑
 $(6-\beta)$;
 between **end**;
 and *epsilon := .5 × 10 ↑ (-d)*;
 Change *lambda [0] := lmin [0] := 1; lambda [1] := omega-eta*;
sum := ro × exp (omega × ro);
 to *lambda [0] := scale; lmin [0] := 1*;
lambda [1] := (omega-eta) × scale;
sum := ro × exp(omega × ro) × scale;
 Change *lmin [L] := Rra [L - 1] × lmin [L - 1]*;
 to **begin**
t1 := Rra [L - 1] × lmin [L - 1];
comment The following constant $5 \uparrow (-10)$ is approximately
 $2 \times \text{base} \uparrow \alpha/\text{scale}$, where *base* is the base of the floating-
 point number system and $\alpha \leq \text{exp} \leq \beta$;
lmin[L] := if abs(t1) > 5 ↑ (-10) then
t1 else 0
end;
 Change *lam [0] := -r1; lam [1] := 1*;
 to *lam [0] := -r1 × scale; lam [1] := scale*;
 Change *lambda [L] := lmin [L] + t1 × (lam [L] + r1 × lmin [L])*
 to *lambda [L] := lmin [L] × scale + t1 ×*
(lam [L] + r1 × scale × lmin [L])
 Change *F[0] := sum/(1+s)*;
 to *F[0] := sum/(scale+s)*;
 The authors gratefully acknowledge the referee's helpful sug-
 gestions.

The policy concerning the contributions of algorithms to
Communications of the ACM has been revised and was pub-
 lished in the August 1970 issue, page 513. Copies of "Algo-
 rithm Policy / Revised August 1970" will be mailed upon
 request.

Short Communications

PROGRAMMING TECHNIQUES

Comment on Bell's Quadratic Quotient Method for Hash Code Searching

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Key Words and Phrases: hashing, hash code, scatter storage,
 calculated address, clustering, search, symbol table, keys, table
 look-up

CR Categories: 3.74, 4.9

In a recent paper [1], James R. Bell gave a method for
 resolving collisions in a hash coded table search which
 generalized the quadratic search method of W. D. Maurer
 [2]. However, he ignored an important anomaly of Maurer's
 method, as well as one singular case.

In the quadratic search method, table locations
 $k + ai + bi^2$ modulo p are examined sequentially for
 $i = 0, 1, 2, \dots$; where p is the table size which is a prime
 number, k is the hash code of the entry's key, and a and b
 are constants with $b \not\equiv 0(p)$. The same table location is
 examined for $i = i_1$ and $i = i_2$, $i_1 \neq i_2$, if and only if

$$k + ai_1 + bi_1^2 \equiv k + ai_2 + bi_2^2 (p)$$

which is true if and only if

$$(i_2 - i_1)[a + b(i_1 + i_2)] \equiv 0.$$

This in turn is true if and only if

$$i_1 \equiv i_2 \text{ or } i_1 + i_2 \equiv (p - a)/_p b,$$

where $/_p$ denotes division in the field of integers modulo p .
 It is easy to see from this that the procedure examines
 $(p + 1)/2$ different table locations, if $p > 2$. However, it
 will examine them all on its first $(p + 1)/2$ tries only if
 $a \equiv 0$. Therefore, for an efficient procedure we should take
 $a \equiv 0$. The search can then be stopped after $(p + 1)/2$
 tries.

Bell's improvement of this procedure consists of letting
 a and b be pseudorandom functions of the entry's key.
 For his algorithm [1, p. 108],

$$a \equiv \text{"some fixed constant"} + Q/_p 2$$

$$b \equiv Q/_p 2,$$

where Q is a function of the entry's key, and $p > 2$. (Al-
 though Bell states that a is a constant, this is not true for
 the algorithm he describes.) The algorithm considers the
 search a failure (table "full" and entry not in it) when it
 returns to the first location examined. This occurs on the
 i th try, where $1 < i \leq p + 1$, $i - 1 \equiv (p - a)/_p b$. At
 that time, every other location which it has examined will

have been examined twice. Clearly, half the table will be searched only if we replace the "fixed constant" by a number congruent to $-Q/p$. However, even if this is done, there is still the problem that when $Q \equiv 0$, only one table location is examined!

To correct these problems, replace steps (3) and (4) of Bell's algorithm with:

- (3) Initialize A with C , where C is defined below.
- (4) Increment A by $2Q$.

For this algorithm, we have $a = Q + C$, $b = Q$. We must then choose C so that $C \equiv -Q$ if $Q \not\equiv 0$ and $C \not\equiv -Q$ if $Q \equiv 0$. The algorithm will then search $(p + 1)/2$ locations if $Q \not\equiv 0$, and will search all p locations if $Q \equiv 0$.

The trouble with this algorithm is that it requires testing for $Q \equiv 0$, which means performing an extra division. A seemingly possible way out is to observe that if $(p - a)/p b \equiv -j$, $b \not\equiv 0$, then the algorithm searches j fewer locations before it starts re-examining locations. We can then try to choose C so that we get j to be small, thereby examining nearly half the table before repeating. However, this requires that we make $C \equiv -(j + 1)Q$. There does not appear to be any simple algorithm for choosing a C satisfying this congruence for a small j when $Q \not\equiv 0$, and choosing $C \not\equiv -Q$ when $Q \equiv 0$.¹ It seems that the division is necessary.

The corrected version of Bell's algorithm still contains a gross inefficiency. For $Q \not\equiv 0$, it decided that the search is a failure after p tries, instead of the necessary $(p + 1)/2$ tries. This is easily corrected by changing the criterion for failure.

In summary, Bell's algorithm requires a correction which adds an extra division to the initialization procedure. This must be considered in evaluating its efficiency. Bell's table comparing the efficiency of his method with that of Maurer's indicates that this extra initialization cost is justified only if checking a single entry is a relatively time consuming operation.

REFERENCES:

1. BELL, JAMES R. The quadratic quotient method: a hash code eliminating secondary clustering. *Comm. ACM* 13, 2 (Feb. 1970), 107-109.
2. MAURER, W. D. An improved hash code for scatter storage. *Comm. ACM* 11, 1 (Jan. 1968), 35-38.

REPLY BY BELL. Before discussing Lamport's comment in detail, let us consider the correct observation on which it is based: Although any quadratic search (including quadratic quotient) hits half of the table entries, sometimes some entries are hit twice before others are hit once.

In other words, $K + ai + bi^2$ may not have maximum period for an arbitrary a and b . The author proves that forcing a to zero will guarantee maximal period.

A much simpler constraint is to let the constant in step

¹ Note added in proof: In his reply below, Bell gives a simple method of choosing $j = 1$.

(3) of the original algorithm be zero. Then

$$h_i(K) = R + (Q/2)i + (Q/2)i^2$$

and we first return to our original hash address when

$$R = R + (Q/2)i + (Q/2)i^2,$$

that is, when

$$i = -1 \quad \text{or} \quad i = 0 \quad \text{or} \quad Q = 0.$$

The first two cases state that $h(K)$ has a maximum periodicity. The third case is the degenerate one where the quotient is congruent to zero. We could use a division to spot the degenerate case. But by adopting the suggestion of paragraph 3 of Section 3c of the original article we can use

$$(Q \wedge \text{lowbitmask}) + 1$$

in lieu of Q to guarantee that this case does not occur.

Lamport has taken a more complicated approach.

SCIENTIFIC APPLICATIONS

On the Number of Automorphisms of a Singly Generated Automaton

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Key Words and Phrases: automata, finite automata, singly generated automata, automorphisms, generators, length of state, minimal-length generators, orbit.

CR CATEGORY: 5.22

1. Introduction

Weeg proved in [3] that the number of automorphisms of a strongly connected finite automaton divides the number of states of the automaton. In [1, Th. 6], the author generalized this result to finite singly generated automata by proving that the number of automorphisms of such an automaton A divides the number of generators of A . This brief note improves the latter result. The number of automorphisms of A is shown to divide the number of minimal-length generators of A .

The improvement is of practical value not only in the more general case of singly generated automata but also in the strongly connected case, for the number of states whose length is minimal is usually much smaller than the number of states of the automaton. The improvement is particularly striking when the number of generators (states, in the strongly connected case) is large but only one of them is of minimal length; in that case, the only automorphism is the identity. But without the present result it may be necessary to examine up to half the number of

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