

AN EXTENSION OF A THEOREM OF HAMADA ON THE CAUCHY PROBLEM WITH SINGULAR DATA¹

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Introduction. Hamada [1] proved the following result about the propagation of singularities in the Cauchy problem for an analytic linear partial differential operator. Assume that the initial data are analytic at the point $\mathbf{0}$ except for singularities along a submanifold T of the initial surface containing $\mathbf{0}$. Let $K^{(1)}, \dots, K^{(m)}$ be the characteristic surfaces of the operator emanating from T . Under the assumption that the $K^{(i)}$ have multiplicity one, he showed that the solution of the Cauchy problem is analytic at $\mathbf{0}$ except for logarithmic singularities along the $K^{(i)}$. We extend his result to the case where the $K^{(i)}$ have constant multiplicity.

1. Definitions and theorem. Let \mathbf{C}^{n+1} denote the set of $(n+1)$ -tuples $\mathbf{x} = (x^0, \dots, x^n)$ of complex numbers. Let S be an n -dimensional analytic submanifold of \mathbf{C}^{n+1} , and let T be an $(n-1)$ -dimensional analytic submanifold of S . Since our results are local, we can assume $S = \{(0, x^1, \dots, x^n) \in \mathbf{C}^{n+1}\}$ and $T = \{(0, 0, x^2, \dots, x^n) \in \mathbf{C}^{n+1}\}$.

Let $D_i = \partial/\partial x^i$, $\mathbf{D} = (D_0, \dots, D_n)$, and let $a: \mathbf{x} \rightarrow a(\mathbf{x}; \mathbf{D})$ be an analytic partial differential operator on a neighborhood of $\mathbf{0}$ in \mathbf{C}^{n+1} . Let $h(\mathbf{x}; \mathbf{D})$ be the principal part of $a(\mathbf{x}; \mathbf{D})$. We assume that S is not a characteristic surface of a at $\mathbf{0}$, so $h(\mathbf{0}; 1, 0, \dots, 0) \neq 0$. Let $\mathbf{p} = (p_0, \dots, p_n)$ be an $(n+1)$ -tuple of formal variables, so $h(\mathbf{x}; \mathbf{p})$ is a homogeneous polynomial in \mathbf{p} with analytic coefficients.

We say that the operator a has *constant multiplicity* at $\mathbf{0}$ in the direction of T if we can factor h as

$$h(\mathbf{x}; \mathbf{p}) = [h_1(\mathbf{x}; \mathbf{p})]^{k_1} \cdots [h_s(\mathbf{x}; \mathbf{p})]^{k_s}$$

for all \mathbf{x} in a neighborhood of $\mathbf{0}$, where each $h_i(\mathbf{x}; \mathbf{p})$ is a polynomial in \mathbf{p} of degree m_i with analytic coefficients, and the Σm_i roots of the polynomials $h_i(\mathbf{0}; \tau, 1, 0, \dots, 0)$ in τ are all distinct. If $s = k_1 = 1$, then a is said to be of *multiplicity one* at $\mathbf{0}$ in the direction of T .

Assume now that a has constant multiplicity at $\mathbf{0}$ in the direction of T . It can be shown that we can find $m = \Sigma m_i$ analytic *characteristic functions* $\varphi^{(1)}, \dots, \varphi^{(m)}$ of h defined in a neighborhood N of $\mathbf{0}$ satisfying:

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1. $h(\mathbf{x}; D \varphi^{(i)}(\mathbf{x})) = 0$ for all $\mathbf{x} \in N$.
2. $\varphi^{(i)}(0, x^1, \dots, x^n) = x^1$ for all $(0, x^1, \dots, x^n) \in N \cap S$.
3. For each $\mathbf{y} \in N \cap S$, the m numbers $D_0 \varphi^{(i)}(\mathbf{y})$ are distinct.

Note that this implies that the numbers $D_0 \varphi^{(i)}(\mathbf{y})$ are the distinct roots of the polynomials $h(\mathbf{y}; \tau, 1, 0, \dots, 0)$ for each $\mathbf{y} \in N \cap S$. Let $K^{(i)} = \{\mathbf{x} : \varphi^{(i)}(\mathbf{x}) = 0\}$, so each $K^{(i)}$ is a characteristic surface of a .

Using these notations, we now state our result.

THEOREM. *Let $a, N, S, T, \varphi^{(i)}$ and $K^{(i)}$ be as above. Let v be an analytic function on N , and let w^j be an analytic function on $N \cap (S - T)$ for $j = 0, \dots, r - 1$, where r is the degree of the operator a . Then there exists a neighborhood U of $\mathbf{0}$ such that the Cauchy problem*

- (1) $a(\mathbf{x}; D)u(\mathbf{x}) = v(\mathbf{x}), \quad (D_0)^j u(\mathbf{y}) = w^j(\mathbf{y}), \quad \text{for } \mathbf{y} \in S, j = 0, \dots, r - 1,$
has a solution u of the form

$$u(\mathbf{x}) = \sum_{i=1}^m F^{(i)}(\mathbf{x}) + G^{(i)}(\mathbf{x}) \log [\varphi^{(i)}(\mathbf{x})],$$

where each $F^{(i)}$ is analytic on $U - K^{(i)}$ and each $G^{(i)}$ is analytic on U .

Hamada proved this result when a has multiplicity one. In this case, if each w^j has at most a polar singularity along T , then each $F^{(i)}$ has at most a polar singularity along $K^{(i)}$. This is false in the general case, as is shown by the solution

$$u(t, y) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \frac{t^{2k+1}}{y^{k+1}}$$

of the two-dimensional Cauchy problem

$$\frac{\partial^2 u}{\partial t^2}(t, y) - \frac{\partial u}{\partial y}(t, y) = 0, \quad u(0, y) = 0, \quad \frac{\partial u}{\partial t}(0, y) = \frac{1}{y}.$$

2. Method of proof. The problem is easily reduced to solving the Cauchy problem (1) with each $w^j \equiv 0$ and v analytic on $N - K^{(1)}$. It can be shown that we may also assume that $h(\mathbf{x}; \mathbf{p}) = h_1(\mathbf{x}; \mathbf{p}) \cdots h_s(\mathbf{x}; \mathbf{p})$, where each h_i has multiplicity one in the direction of T and has $\varphi^{(1)}, \dots, \varphi^{(m)}$ as characteristic functions (so $r = ms$).

Let the functions f_k be the ones defined by Hamada satisfying $df_k/dt = f_{k-1}$, for all integers k , and $f_0(t) = \log t$. The first step is to show that there exists a neighborhood V of $\mathbf{0}$ such that if v is of the form

(2)
$$v(\mathbf{x}) = \sum_{i=1}^m \sum_{k=0}^{\infty} v_k^{(i)}(\mathbf{x}) f_{k-i} [\varphi^{(i)}(\mathbf{x})],$$

with each $v_k^{(i)}$ analytic on V , then the Cauchy problem

$$h_l(\mathbf{x}; \mathbf{D})u(\mathbf{x}) = v(\mathbf{x}), \quad (D_0)^j u(\mathbf{y}) = 0, \quad \text{for } \mathbf{y} \in S, j = 0, \dots, m - 1,$$

has a formal series solution of the form

$$u(\mathbf{x}) = \sum_{i=1}^m \sum_{k=0}^{\infty} u_k^{(i)}(\mathbf{x}) f_{k-l+m-1} [\varphi^{(i)}(\mathbf{x})],$$

with each $u_k^{(i)}$ analytic on V . Moreover, bounds are obtained for the partial derivatives of the $u_k^{(i)}$ in terms of those of the $v_k^{(i)}$. This procedure is similar to the one used by Hamada.

Employing this result s times shows that with v given by (2), the Cauchy problem

$$h_1(\mathbf{x}; \mathbf{D}) \cdots h_s(\mathbf{x}; \mathbf{D})u(\mathbf{x}) = v(\mathbf{x}), \quad (D_0)^j u(\mathbf{y}) = 0, \quad \text{for } \mathbf{y} \in S, j = 0, \dots, r - 1,$$

has a formal solution

$$u(\mathbf{x}) = \sum_{i=1}^m \sum_{k=0}^{\infty} u_k^{(i)}(\mathbf{x}) f_{k-l+r-s} [\varphi^{(i)}(\mathbf{x})]$$

with the $u_k^{(i)}$ analytic on V . Again, bounds are obtained on the $u_k^{(i)}$.

Now we write $a(\mathbf{x}; \mathbf{D}) = h_1(\mathbf{x}; \mathbf{D}) \cdots h_s(\mathbf{x}; \mathbf{D}) + b(\mathbf{x}; \mathbf{D})$, where the degree of b is less than r . Using the above results, we solve the sequence of Cauchy problems

$$h_1(\mathbf{x}; \mathbf{D}) \cdots h_s(\mathbf{x}; \mathbf{D})_q u(\mathbf{x}) = \begin{cases} v(\mathbf{x}) & \text{if } q = 0, \\ -b(\mathbf{x}; \mathbf{D})_{q-1} u(\mathbf{x}) & \text{if } q > 0. \end{cases}$$

$$(D_0)^j_q u(\mathbf{y}) = 0, \quad \text{for } \mathbf{y} \in S, j = 0, \dots, r - 1,$$

to get

$$(3) \quad q u(\mathbf{x}) = \sum_{i=1}^m \sum_{k=0}^{\infty} q u_k^{(i)}(\mathbf{x}) f_{k-l-q(s-1)} [\varphi^{(i)}(\mathbf{x})]$$

with each $q u_k^{(i)}$ analytic on V . Then

$$(4) \quad u(\mathbf{x}) = \sum_{q=0}^{\infty} q u(\mathbf{x})$$

is easily seen to be a formal solution of (1) (with $w^j \equiv 0$).

Now assume $v(\mathbf{x}) = v_l(\mathbf{x}) f_{-l} [\varphi^{(1)}(\mathbf{x})]$, with v_l analytic on V , and let the corresponding solution (4) be $u_l(\mathbf{x}) = \sum_{i=1}^m u_i^{(i)}(\mathbf{x})$. Using the bounds on the $q u_k^{(i)}$, we can find a neighborhood W of $\mathbf{0}$ and demonstrate the absolute convergence of the sums (3) and (4) to prove that $u_l^{(i)}$ is analytic on $W - K^{(i)}$. Furthermore, we obtain a bound on $u_l^{(i)}$ in terms of a bound on v_l .

Finally, we can write $v(\mathbf{x}) = \sum_{l=1}^{\infty} v_l(\mathbf{x}) f_{-l} [\varphi^{(1)}(\mathbf{x})]$ (plus an analytic term which is handled by the Cauchy-Kowalewski theorem). It can be shown that there is a neighborhood U of $\mathbf{0}$ such that the sums $u^{(i)}(\mathbf{x}) = \sum_{l=1}^{\infty} u_l^{(i)}(\mathbf{x})$

are absolutely convergent on $U - K^{(i)}$. It is then easily seen that the solution $u(x) = \sum_{i=1}^m u^{(i)}(x)$ has the desired form.

3. Further generalizations. It is evident from the proof that the theorem remains valid if v has a singularity along any of the hypersurfaces $K^{(i)}$. The theorem is also true if v has a singularity on any hypersurface K containing T which is not tangent to S or to any $K^{(i)}$ at 0 .

By using different choices for the functions f_k , the result can be extended to the case where the w^j are p -valued analytic functions on $N \cap (S - T)$ —i.e., multiple-valued functions finitely ramified about T —and v is a p -valued analytic function on $N - K^{(i)}$ or $N - K$. In this case, the $F^{(i)}$ become p -valued analytic functions on $U - K^{(i)}$. This result was also obtained by Wagschal [2] when a has multiplicity one.

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