MATH BULLETIN

1957

BOARD OF EDUCATION OF THE CITY OF NEW YORK
THE BRONX
HIGH SCHOOL OF SCIENCE
120 EAST 184 STREET NEW YORK 68, N. Y.

Robert Teitel

MATHEMATICS BULLETIN
1957

CONTENTS

ACKNOWLEDGEMENTS

The staff wishes to express its gratitude to Dr. Meister; to Dr. I. A. Dodes for his valuable assistance and advice as Chairman of the Department, and to the Student Organization for its help in the publication of the magazine.

Editorial

In the Bronx High School of Science a great variety of mathematics courses is offered to the students. Some of the readers of the Math Bulletin have (or will have) taken three years of math, the barest minimum. Others, at the other extreme, take what is perhaps the maximum amount of mathematics offered in high schools, the sequence leading up to College Math. The rest of the student body takes sequences in between these extremes. Yet, how many Sciencites have ever stopped to wonder why they take Math? How many have ever realized what mathematics *is*, what it really *does*? Probably not many students have. It is therefore the purpose of this *Math Bulletin* to introduce its readers to some of the basic concepts of mathematics.

Mathematics, once only an intellectual plaything, has become a valuable tool of the scientist, and is one of the most exciting and challenging pursuits of mankind. The enormous value to science of mathematics arises because it enables us to use convenient symbols to represent a model of reality. However, we would like to point out an extremely important distinction between *pure* and *applied* mathematics.

In pure mathematics, symbolic form is used entirely as an abstraction. It exists by itself as a "mental exercise," as a sort of "game," apart from the world. Yet, from this highly abstract subject have come results of utmost importance to the development of our concepts of reality. However, fascinating as this topic may be, we would like to present at this point the basic structure of pure mathematics, and ignore its applications.

In this "game", specific groupings of symbols (theorems) are derived from an original set of assumptions known as postulates. If these postulates are called true, then the results of the application of the proper logic methods are just as true. Assumptions are made arbitrarily, having no restrictions imposed upon them except "consistency". Consistency means that no contradiction is inherent in the postulates; that is, no theorem can be proved both true and not true from the same set of postulates.

Another desirable characteristic of a set of postulates is "independence." This means that no postulate can be derived from the others. If a postulate is truly independent of the others in a set of postulates, its opposite may be assumed instead of it and will also be compatible with the other postulates. For example, Euclid's Fifth Postulate says that through a given point one, and only one, line can be drawn parallel to a given line. This postulate is independent of all the others, and so merely by accepting its opposite entirely new systems of geometry have been developed.

In applied mathematics, however, the generalized symbolic form is used for another reason: it is easier to work with. Using mathematics, a scientist can represent a relationship between concrete objects in terms of an identical relationship between abstract symbols (which are often easier to handle). Thus, he makes his work more convenient, since anything shown to be true of the symbols is true of what they represent. In addition, the prejudices we attach to concrete objects are eliminated when we deal with symbols, and the chances of error in interpretation of data are reduced. After all, what we perceive may not be reality at all, but a psychological and physiological delusion. For these reasons, mathematics is necessary to science.

The scientist, however, is under a handicap when using mathematics. He must realize that mathematics can unearth only those relationships included as part of the data being considered, and cannot pull "great truths" out of the air. In addition, the results of mathematical analysis are as approximate as the information originally submitted to the analysis. If the information is restricted, so are the conclusions. Only the factors submitted are taken into account.

The difference between pure and applied mathematics must always be remembered: one is a "game", and the other is an attempt at a representation of reality.

The concepts and techniques of mathematical analysis, the basis of all mathematics, is the subject of this issue of the *Math Bulletin*. We hope we have supplied a pleasant introduction to these essentials.

The P.H.D. Triangle and the Golden Section

By means of the peculiar properties of a certain triangle constructed within a circle it is possible to derive the relationship between an arc of a circle and its chord. Furthermore, these properties enable one to inscribe regular pentagons, hexagons, and decagons without resorting to the use of a protractor. This singular triangle is known as the PHD triangle (Pentagon, Hexagon, Decagon) and the following are the steps in its construction: (See Figure 1)

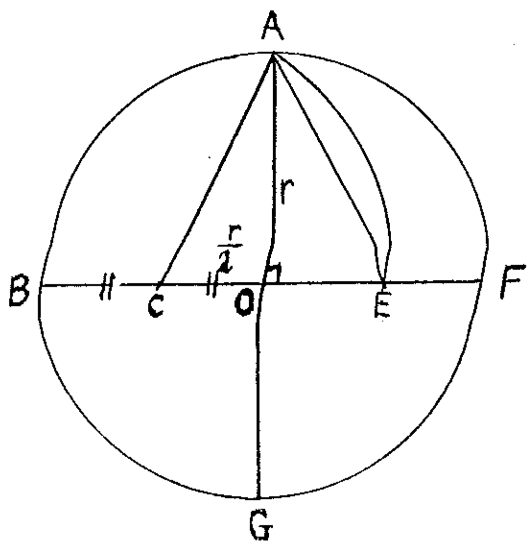


Figure 1

1. Draw circle (0) and diameters BF and AG perpendicular to each other.
2. Divide OB into two equal parts: BC and CO.
3. With C as a center, and AC as a radius, strike arc AE.
4. Triangle AOE is the PHD triangle.
   AE is the side of the inscribed pentagon.
   OE is the side of the inscribed decagon.
   AO is the side of the inscribed hexagon.

Because of space limitations, a formal proof of this construction has been omitted. The reader can find the proof in any textbook.

It is of interest to determine the lengths of the sides of the inscribed polygons in relation to the radius of the circle.

In Figure 1:

1. $CO = r/2$
2. $AO = r$, the side of the inscribed hexagon
3. $AC = \sqrt{\left(\frac{r}{2}\right)^2 + r^2}$
4. $AC = \frac{r\sqrt{5}}{2}$
5. Since $CE = AC$ and $OE = CE - CO$, $OE = \frac{r\sqrt{5}}{2} - \frac{r}{2}$
6. $OE = \frac{r(\sqrt{5} - 1)}{2}$, the side of the inscribed decagon
7. $AE = \sqrt{\frac{r^2}{4}(5 - 2\sqrt{5} + 1) + r^2}$
8. $AE = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}$, the side of the inscribed pentagon

We may now relate the chords of the circle to their arcs.

(a) The arc of a regular pentagon is $\frac{2\pi r}{5}$; the arc chord ratio is:

$$\frac{\frac{2\pi r}{5}}{\frac{r}{2}\sqrt{10 - 2\sqrt{5}}} = \frac{4\pi}{5\sqrt{10 - 2\sqrt{5}}},$$

or approximately 1.068.

(b) The arc-chord ratio for the hexagon is $\pi/3$, or 1.0472.

(c) The arc chord ratio of the decagon is

$$\frac{2\pi}{5(\sqrt{5} - 1)}, \text{ or } 1.0167.$$

Thus, the arc-chord ratio approaches unity as the arc length decreases. Considering upper limits, the ratio of a semicircle to its diameter is $\pi/2$, or 1.5708. Of course, the limiting ratio of a major arc to its chord is infinity, since $2\pi r/0$ equals infinity.

There is also an interesting relationship between the PHD triangle and the

continued on page 9

INTRODUCTION TO GROUPS

Bertrand Russell said, "Pure mathematics consists entirely of such asseverations as that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is true, and not to mention what the anything is of which it is supposed to be true."

Perhaps the most abstract topic in mathematics is that of Groups. It has a great many applications, perhaps as a result of its extreme generality.

These are the essentials of Group theory:

I. A set of undefined "elements," called a, b, c, etc. These may, if we wish, be interpreted any way we please – as numbers, points, canaries, or planets. The enormous value of Group theory rises from the many different interpretations which may be given to these elements. However, for the purposes of Group theory these symbols have no specified meaning.

II. An undefined "operation," called "o." This operation may also be given a meaning, if convenient. This might be "multiply," or "moves in such and such a manner," or "rotates on an axis," etc. It should be emphasized that in Group theory no specific meaning is assigned to the operation.

III. Four essential properties, or postulates:

1. When one element is operated upon by another, the result is an element of the same set. That is:

$$a \text{ o } b = c$$

This is known as the Closure postulate.

2. There exists a unique element, "i," such that:

$$a \text{ o } i = i \text{ o } a = a$$

The placement of "i" (the Identity element) is quite important. A distinction must be made between operating on the "left," and on the "right." This will be treated in No. 5.

3. For every element "a" there exists an element "a'" such that:

$$a \text{ o } a' = a' \text{ o } a = i$$

The element "a'" is called the Inverse of "a."

4. For any three elements a, b, and c, the following is true:

$$(a \text{ o } b) \text{ o } c = a \text{ o } (b \text{ o } c)$$

This is the Associative property, and completes the four essential postulates.

5. In some cases a fifth property, Commutative, is possible:

$$a \text{ o } b = b \text{ o } a$$

In some groups these two operations do not give the same result. When they do, the group is known as an Abelian group.

An excellent example of a group is the integers (both positive and negative), with the operation "addition." They are an Abelian group because it can be shown that the five postulates are satisfied. For example:

$6 + 4 = 10$ (Closure)

$6 + 0 = 0 + 6 = 6$ (Identity)

$6 + (-6) = (-6) + 6 = 0$ (Inverse)

$6 + (3 + 5) = (6 + 3) + 5$ (Associative)

$6 + 3 = 3 + 6$ (Commutative)

Working with the basic postulates, and several others, it is possible to prove many general theorems about the elements. Once proved true of groups in general, they are then *all* automatically true about any set of elements and operation that can be shown to be a group. The additional axioms are:

1. Reflexive (a = a)
2. Symmetric (a = b implies that b = a)
3. Transitive (If a = b and b = c, then a = c)
4. Substitution (An element equal to another may be substituted for it without changing the value of the expression.)

We are now in a position to prove the theorem:

If a = b, then a o c = b o c

1. a o c = a o c (Reflexive)
2. a = b (Given)
3. a o c = b o c (Substitution)

continued on page 7

BRAID THEORY

The fundamental theory of braids incorporates group theory. Therefore, an understanding of the preceding article is essential. First, what is a braid? Braids may be considered as physical entities, or as abstract mathematical forms. We shall consider them as physical entities. Let us now define a braid. (We shall assume an intuitive understanding of the terms used.)

Definitions:

(1) A braid is a configuration of strands. One strand may cross another by going over or under it.

(2) A braid must have two sets of terminals, the top terminals and the bottom terminals.

(3) Each strand must have one terminal on either end of it.

(4) Each strand must be continuous between the two terminals.

(5) The number of top terminals and bottom terminals must equal each other and must equal the number of strands.

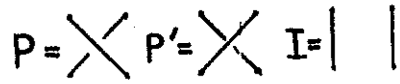
Some braids are shown in the accompanying illustrations. The shaded squares are not part of the braid and will be explained later.

Now that we have defined braids, we may investigate their properties. It can be shown that braids having the same number of strands form a group. The first requirement for a group is a group operation. I define the operation AB (A multiplied by B) as attaching the top terminals of braid B to the bottom terminals of braid A. It can be seen that if A and B are braids of N strands each, then AB is a braid of N strands. Hence, the closure law is satisfied by this operation. An illustration of the multiplication of braids is given in Figure 1. It should be noted that AB does not necessarily equal BA; hence, braids do not form an abelian group under the operation of multiplication.

When I said that AB does not always equal BA, I assumed an understanding of an equality relationship among braids. I shall now define this relationship. Two braids are equal if they can be made to coincide by moving them and by "manipulating" the strands without disconnecting them from their terminals. This "manipulation" may consist of moving and stretching or contracting the strands. Thus, the size of the braid is unimportant. If we find the product CC' of the two braids shown in Figure 2, this product can be shown to be equal to a braid with none of the strands crossing each other. In this way, many braids may be shown to be equivalent to simpler braids. The braid with no strands crossing is simpler than that formed by attaching the terminals of C and C' to each other, since the first has fewer crossings.

The next requirement for a group is that there be an identity element I, so that for any braid A, AI equals A. It can be seen that the identity braid is the braid with no strands crossing. If the terminals of this braid are attached to the bottom terminals of any other braid, it is equivalent to stretching this braid. Since by contracting the strands of this braid it can be made to coincide with the original braid, it must be equal to the original braid. As with any other group, AI = IA = A.

In order for braids to form a group, for every braid A, there must be an inverse braid A' so that AA' equals I, the identity element. C' is the inverse of C in Figure 2 because CC' = I. However, C'C also equals I. This illustrates the postulate, common to groups, that AA' = A'A = I, for any A. The properties of the inverse element are very interesting when applied to actual physical braids, such as strings woven together. If the inverse braid is tied to the woven strings, and the strings are pulled tight, the identity element, the braid with no strings crossing, will be formed. Thus, to untie a pigtail in a girl's hair, merely tie strings to the strands of hair, tie the inverse braid, and pull. Although painful, this method will work.



The last requirement for a group is that the associative law be true. This means that for any braids A, B and C, A(BC) = (AB)C. It can be seen that it does not matter which two sets of terminals are attached first; the end result will be the same.

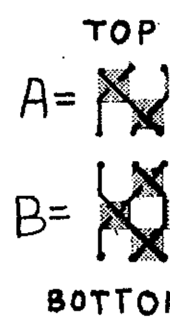
Since braids satisfy the four conditions of a group, they form a group under the operation of multiplication. I must emphasize that only the braids having the

continued on page 7

Braid Th
(continued)

same number
multiplicatio
braids do n
strands. He
form a group
but together
Now that I
a group, I ca
of group the
before, are:

I have pr
the braid. I
it is necess
representatio
The laws of
be useless
formed into th
I decided
tion of braid
arrangement
ments.) The
the fundamen
of P) and I
strands).
To form a
arranged in
these squar
braids with t
of the squa
manner are s
is necessary
the outer verti
Such strands
It is also n
between the
and bottom
strands are n
to make the
the terminals
"b" in Figu
such an opera
any braid can



Braid Theory
(continued)

same number of strands form a group since multiplication is undefined if the two braids do not have the same number of strands. Hence braids of four strands form a group as do braids of five strands, but together they do not form one.

Now that I have shown that braids form a group, I can apply to them the theorems of group theory. Two postulates mentioned before, are:

$$AI = IA$$
$$AA' = A'A.$$

I have presented the physical form of the braid. In order to work with braids, it is necessary to have a mathematical representation of the physical braids. The laws of physics, for example, would be useless if they could not be transformed into the language of algebra.

I decided to use a matrix representation of braids. (A matrix is a rectangular arrangement of rows and columns of elements.) The elements of the matrix are the fundamental braids P, P' (the inverse of P) and I (the identity braid of two strands).

To form a braid, draw a set of squares arranged in a checkerboard pattern. On these squares, place the fundamental braids with the terminals on the vertices of the squares. Braids formed in this manner are shown in Figures 1 and 2. It is necessary to define strands between the outer vertices of the outermost squares. Such strands are marked "a" in Figure 1. It is also necessary to define strands between the outer terminals and the top and bottom outer squares where the strands are not otherwise there, in order to make the strands continuous between the terminals. These strands are marked "b" in Figure 1. It can be shown that such an operation defines a braid and that any braid can be formed in this manner.

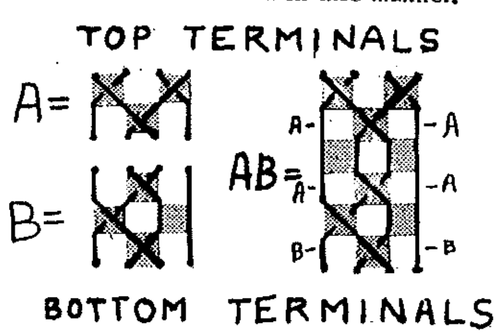


Figure 1

Introduction to Groups
(continued)

Similarly, it can be proved that if a = b, then c o a = c o b. If we were to call our elements numbers, as in the previous illustration, and called the operation multiplication, this theorem would be recognizable as the technique of multiplying both sides of an equation by the same thing. This technique is basic to most algebraic manipulations. A corollary of this is that if a = b and d = c, then a o c = b o d. You might try to prove this, using the previous theorem.

It can also be proved that an equation of the type a o x = b has only one solution, for specific values of a and b. This is another example of an "intuitive" principle of elementary algebra which is really a very simple consequence of Group theory. In fact, all the laws we are taught in arithmetic and algebra are found in Group theory, since the real numbers form a group under addition and multiplication.

The theorems and applications of Group theory are much greater than I have indicated here. Group theory is finding a widespread use in the newer fields of mathematics, such as topology. It is one of the most rapidly growing fields at a time when mathematics is in a state of expansion and discovery. In fact, topology has been defined as "the study of the invariants of geometric configurations under groups of transformations."

Wherever groups have been introduced in mathematics they have tended to connect previously separated subjects, and to unify and clarify all mathematics. If anyone has been under the impression that mathematics is a stale and stagnant subject, with nothing new to discover, one glance into Group theory ought to change his mind!

Alison Lord

Since the method of formation of the braid has been shown, the mathematical representation is simple. One must merely insert the symbol of the element in its relative position in the matrix. Hence, in Figure 1:

continued on page 12

TRIANGULAR COORDINATES

We are all familiar with the system of rectangular Cartesian coordinates, which locates any point on a plane by giving its distances from two perpendicular axes. There is, however, another coordinate system, which locates a point by giving its distances from two fixed reference points.

The following conventions are followed in this system: the distance between the fixed points (X and Y) is equal to one unit. The distance between any two points is always positive. All points whose x-coordinates are equal to c lie on the circle with x as center and radius c.

Figure 1 shows the location of $x = 1$, $y = 1$. Notice that there are two different points satisfying these coordinates, since the unit circles around X and Y intersect in two points. In fact, all points except those on the horizontal line bisecting the plane have twins.

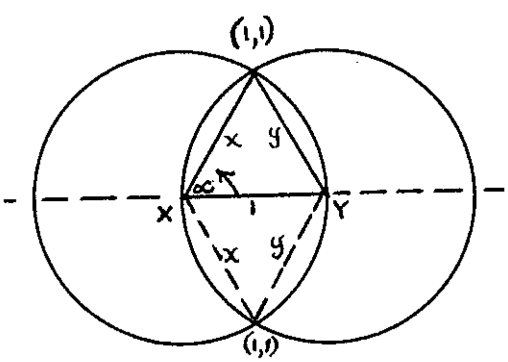


Figure 1

Let us now consider the equations of certain familiar loci. An ellipse is defined as the locus of all points, the sum of whose distances from two fixed points is constant. Therefore, the graph of the equation $x + y = k$ is an ellipse with foci X and Y, provided $k>1$. If $k = 1$, the graph is the straight line connecting the X and Y points. If $k<1$, there is no graph.

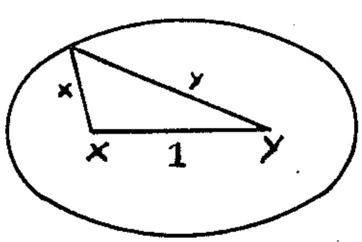


Figure 2

A hyperbola is the locus of points such that the difference of its distances from two fixed points is a constant. The equation of a hyperbola is $x-y = k$, or $y-x = k$, one branch being given by each formula. Since neither coordinate can exceed the other by more than one, k must be less than one. If $k = 1$, we have the extension of the axis.

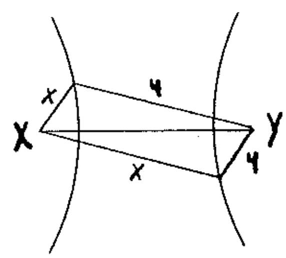


Figure 3

A parabola is defined as the locus of all points equidistant from a fixed line and a fixed point. If we let the focus be the X point, and the directrix be the line AB perpendicular to the axis at a distance c from the X point, we get the fourth figure, $XM = c$, and $EY = 1-(c-x)$. Choosing a point F on the parabola,

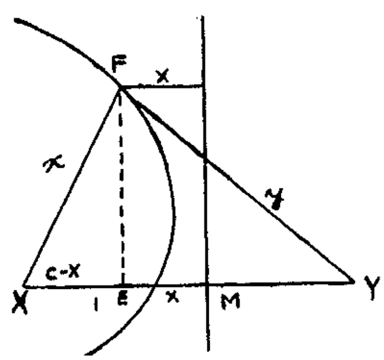


Figure 4

$$XF = FG = EM = x;$$

then,

$$\overline{EF}^2 + 1-2(c-x) + (c-x)^2 = y^2,$$
$$\overline{EF}^2 = x^2 - (c-x)^2 \text{ ; by substi-}$$

tution,

$$x^2 + 2x + (1-2c) = y^2$$

is the equation of such a parabola. As

continued on page 9

of points such
istances from
nt. The equa-
k, or y-x = k,
each formula.
n exceed the
must be less
the extension

; the locus of
. a fixed line
t the focus be
ix be the line
s at a distance
e fourth figure,
oosing a point

M = x;

) + (c-x)^2 = y^2,
-x)^2 ; by substi-

2c) = y^2
a parabola. As

ge 9

Triangular Coordinates
(continued)

examples, if the directrix is the perpendicular bisector of the XY-axis, then $c = \frac{1}{2}$, and the parabola becomes $x^2 + 2x = y^2$; if AB passes through Y, $c = 1$, and the equation is $x^2 + 2x - 1 = y^2$.

Straight lines are somewhat more difficult in triangular coordinates than in rectangular ones. However, referring to our triangle in the first figure, we see that, by the law of cosines,

$$y^2 = x^2 - 2x \cos a + 1.$$

If we plot a straight line through the X point at angle a, then cos a is constant, which is designated k. Therefore,

$$y^2 = x^2 - 2kx + 1$$

is the equation of such a line, provided that k is between +1 and -1. Note that if $k = \pm 1$, $a = 0°$ or $180°$, and we have the equations of the extensions of the axis, as expected:

$$y^2 = x^2 \pm 2x + 1 = (x \pm 1)^2; \; y = x \pm 1.$$

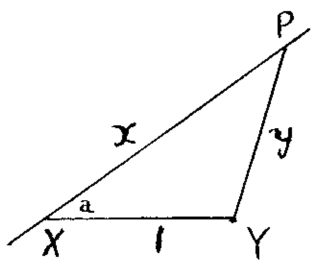


Figure 5

The equation of a line perpendicular to the axis at a distance c from the X point may be determined by drawing a right triangle, as in figure six:

$z^2 = x^2 - c^2$, but $z^2 + (c+1)^2 = y^2$, so,
$y^2 = x^2 + 2c + 1$.

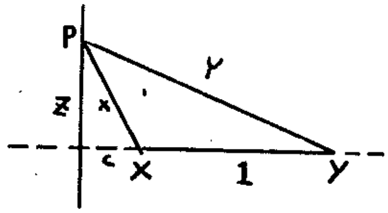


Figure 6

Note that when c = '0', we have the equation of a line perpendicular to the axis through the X point, $y^2 = x^2 + 1$. This is the same as passing a line through X with angle $a = 90°$; the cosine of "a" then equals 0, and we obtain the same equation.

You may wish to investigate the effect of varying the distance between the X and Y points, or to determine the equations of other loci.

Problems

1. What type of symmetry does the system illustrate? Why?
2. Prove that neither coordinate may exceed the other by more than 1. (Hint: Consider a circle of radius c about the X point, and another of radius c-2 about Y.) Also, prove that the sum of the coordinates must be equal to, or greater than, 1. (Hint: Use circles of radii k and 1-k about X and Y.)
3. Apply the results of problem 2 to equations of ellipses and hyperbolas.
4. Derive an equation of a line inclined at an angle a to the axis, and at a distance c from X.

Arthur Millman

P.H.D. TRIANGLE
(continued)

golden section. A line is divided into the golden section if the ratio of its larger part to the whole equals the ratio of the smaller part to the larger part. If the whole line be equal to unity, and the larger part equal to x, the smaller part equals 1 – x, and, by definition of the golden section,

$$\frac{x}{1} = \frac{1-x}{x}, \text{ or } x^2 = 1 - x.$$

Solving the quadratic equation by the quadratic formula, $x = (\sqrt{5} - 1)/2$, or approximately .618. Since OE equals:

$$\frac{r(\sqrt{5}-1)}{2},$$

as shown above, OE equals the larger segment of the golden section of the radius. Thus, the PHD triangle is a means of constructing the golden section.

For those of you who have not had enough of the PHD triangle and extreme and mean ratio, I suggest consulting *What is Mathematics,* edited by James R. Newman, and *Mathematics for the Millions,* by Hogben (the latter for data on the Fibonacci Series).

MARK BRIDGER & TODD GITLIN

CONIC SECTIONS

Ever since René Descartes invented analytic geometry in the seventeenth century, men have puzzled over the geometric counterparts of algebraic equations. One such equation is the general quadratic equation in two unknowns,

$$(1)\ Ax^2+Bxy+Cy^2+Dx+Ey+F=0,$$

in which A, B, C, D, E, and F are constants. It has been shown that when an equation of this form is graphed in normal rectangular coordinates, the resulting figure could have been obtained by slicing a cone.

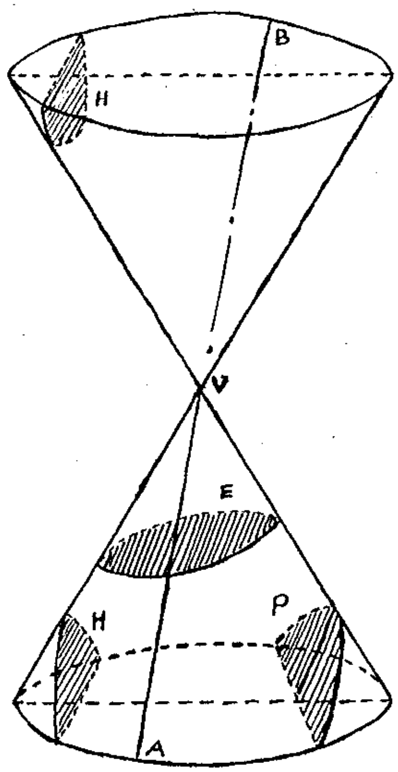


Figure 1

Before proceeding, we must define our terms. A right circular cone, the only kind we shall discuss, is formed by rotating a right triangle about one of its legs. The surface thus generated is one nappe of a true cone. Strictly speaking, the hypotenuse of the triangle must be extended indefinitely in both directions before it is revolved. The resulting surface is somewhat like an hourglass, or two ice cream cones placed tip to tip. The vertex of a cone is the point at which the nappes terminate. An element is any straight line that lies wholly on the surface of the cone, and passes through the vertex. The axis of the cone is the leg about which the generating right triangle has been revolved. A conic section is defined as the intersection of a plane and a true cone.

An ellipse is the conic formed by slicing one nappe of a cone so that the cutting plane intersects all of the elements. It is important as the shape of the orbit of each of the planets, and is of architectural significance. If we choose the proper axes, the equation of the ellipse is

$$(2)\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

A hyperbola is formed when a plane cuts both nappes of a cone, and is parallel to two elements. It has two distinct branches. It has several military uses, among them long range navigation. The equation of a hyperbola is

$$(3)\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Note the similarity of the equations of the ellipse and hyperbola. As an exercise, pick values of a and b, and try plotting both the ellipse and the hyperbola. Refer to the article on triangular coordinates, and see if you can find the foci of the two curves.

The parabola is formed when a plane intersects a cone so that it is parallel to one and only one element. It cuts only one nappe of the cone. It is the path followed by projectiles, and has optical uses. Its equation is

$$(4)\ y^2=4px.$$

Through the proper choice of axes, it is always possible to transform an equation of form (1) to one of the other three forms. Even more interesting is the study of the degenerate loci formed when the slicing plane assumes a limiting position. For instance, if the plane of an ellipse is cut perpendicular to the axis, we get a circle. In that case, the foci coincide, and the equation becomes $x^2+y^2=a^2$, since $a=b$. It is possible to make the plane of the parabola approach the element to which

continued on page 12

A True-False Logic System

Logic is a subject which readily lends itself to mathematical analysis. In this article we shall explain the elements of the two-valued logic system, in preparation for the more advanced article which follows.

The basis of all logic systems is the "proposition." We may postulate that every proposition is a statement which can be assigned an arbitrary value of "truth" or "falsity." A statement which is neither true nor false is not, in this system, a proposition. A logic system may contain only those words that have been previously defined or classified as undefined terms. Since our language is positively oriented, we do not usually qualify our positive, or true, statements. We say, "It is raining," not, "It is true that it is raining." Our negative (false) propositions are always qualified with "not," or, more formally, "It is false that...." As our language, our modern symbolic logic does not contain a qualifier for truth.

Two important types of propositions are postulates and theorems. Postulates are propositions whose qualifiers are determined by assumption. It is necessary to remember that, in logic, a postulate is only true if we say so; thus, logic systems may not relate directly to experience. In fact, many interesting systems are without intrinsic meaning. For example, if we say, "Up is down," and base a consistent logic system on this statement, no one can say that "up" really isn't "down," experience playing no part in the matter. For the purpose of our logic system, "up is down," because we say so.

Theorems are statements that are proved, as in geometry, by formalized methods, using only postulates and previously proved theorems. A theorem is defined as a proposition that can be logically inferred from a given set of postulates and definitions.

What makes mathematical logic so interesting, and at the same time divorces it from the pitfalls of language, is the exclusive use of symbols. Propositions are indicated by small letters p, q, r, etc. There are also symbols for certain operations in a logic system. "Λ" means "and"; the result of combining two propositions in this way is called their conjunction. For instance, $p \Lambda q$ is a short way of writing "p is true and q is true." The symbol "V" means "or"; $p V q$ is called the disjunction of p and q. "$p \to q$" is an implication, and means "if p is true, then q is true." The equivalence relationship is expressed by the symbol "$\leftrightarrow$." It is a sort of double implication, and "$p \leftrightarrow q$" means "if p is true, then q is true AND if q is true, then p is true." Finally, there is a symbol, "$\sim$," for negation. "$\sim p$" means "it is false that p is true."

The most important postulate of the logic system is that the result of performing any of the above operations is also a proposition; in other words, "$p V q$" and "$p \to [(q \Lambda r) \leftrightarrow (\sim s)]$," or any other such set of symbols, is a proposition which may be assigned a logical value. Our next postulates, expressed as a truth table, tell us what value to assign the result of each individual operation, depending upon the values of its component elements.

	p	$\sim p$	q	$p \Lambda q$	$p V q$	$p \to q$	$p \leftrightarrow q$
(1)	T	F	T	T	T	T	T
(2)	T	F	F	F	T	F	F
(3)	F	T	T	F	T	T	F
(4)	F	T	F	F	F	T	T

A tautology is defined as a complex proposition whose truth is independent of the truth or falsity of its component elements. Truth tables are very useful in the proof of tautologies, as we can list all of the possibilities and see if the proposition in which we are interested is always true. As an example, we prove that $(p \to q) \leftrightarrow [(\sim p) V q]$ is a tautology:

p	q	$p \to q$	$\sim p$	$(\sim p) V q$	$(p \to q) \leftrightarrow [(\sim p) V q]$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since the last column, which is the theorem we wanted to prove, is always true, it is a tautology. The reader should work out the meaning of the tautology for himself.

The use of quantifiers, as well as qualifiers, is necessary in logic. If we use qualifier "It is true that..." in the sentence, "It is true that men have black hair," we have said something absurd. If, however, we use the quantifier "some," and say, "It is true that *some* men have

continued on page 12

Conic Sections

(continued)

it is parallel, finally becoming tangent to the cone. The equation then becomes $y^2 = 0$, since $x = 0$. If the plane of a hyperbola passes through the vertex, then the locus degenerates into a pair of intersecting straight lines. If an ellipse is moved parallel to itself until it passes through the vertex, it becomes a point. A comparison with degenerate forms of the conics, as expressed in triangular coordinates, is suggested.

Anyone who wishes to investigate the properties of the general quadratic in two unknowns, as well as the special cases illustrated here, is referred to any good textbook of analytic geometry. For those who doubt that the conditions expressed here and those in the article on triangular coordinates define the same loci, a book such as Wilson and Tracey's *Analytic Geometry* will prove quite conclusively their similarity.

If an ice-cream cone, resting on its base, may be sliced vertically only, which conic sections could be formed? Why can't the others be formed? What about horizontal slices?

ARTHUR BASS

True-False Logic

(continued)

black hair," we are speaking correctly. The logical symbol for "some" is $\exists x$, which means, "There is at least one x such that...." Note that "some" means at least one. Many authors do not quantify "all," but the use of $\forall x$ to mean "for all x" is replacing the convention of placing (x) before a proposition.

Negations in logic conform to the "economy principle." The negation of $\forall x(p)$ is not $\forall x(\sim p)$, but $\exists x(\sim p)$. The existence of even one x for which p is false is enough to negate $\forall x(p)$, which states that p is true for *all* x. In these cases it is taken for granted that p is a proposition involving x. The negations of other common forms, which may be easily verified by the reader, using truth tables, follow:

$\sim(p \wedge q) \leftrightarrow [(\sim p) \vee (\sim q)]$

$\sim(p \vee q) \leftrightarrow [(\sim p) \wedge (\sim q)]$

$\sim(\sim p) \leftrightarrow (p)$

$\sim(p \rightarrow q) \leftrightarrow [p \wedge (\sim q)]$

$\sim(p \leftrightarrow q) \leftrightarrow [p \leftrightarrow (\sim q)]$

$\sim[\exists x(p)] \leftrightarrow \forall x(\sim p)$

There are three implications related to

continued on page 14

Braid Theory

(continued)

$$A = \left|\begin{array}{ccc} P & & P' \\ & P & \end{array}\right| \qquad B = \left|\begin{array}{ccc} & P & \\ P & & I \\ & P & \end{array}\right| \qquad AB = \left|\begin{array}{ccc} P & & P' \\ & P & \\ I & & I \\ & P & \\ P & & I \\ & P & \end{array}\right|$$

The reader will note that if the top terminals of B are attached to the bottom terminals of A in the operation AB, the terminals will not form the checkerboard squares. Therefore, a row of identity elements is inserted between them so that the operation becomes AIB, which is equal to AB.

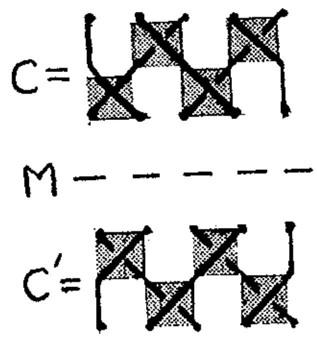


Figure 2

It can be shown that, to form the inverse braid, we use the mirror image as before, but we replace every element by its inverse; $(P)' = P'$, $(P')' = P$, $(I)' = I$. The braids illustrated in Figure 2 are:

$$C = \left|\begin{array}{cccc} & P & & P \\ P & & P & \end{array}\right| \qquad C' = \left|\begin{array}{cccc} P' & & P' & \\ & P' & & P' \end{array}\right|$$

LESLIE LAMPORT

A two-value
simplicity for
teresting, non
of including a
whose truth o
"I don't kno
that the proc
depends upon
a proposition
ascertained.
effect of this
our hypothesi
assign this i
and develop
third value. I
prove that e
pressed as t
proof of this
upon the foll
number can
two primes.
has never be
Adapting a
this problem,
determinate
Then we co
our odd numb

We set u
with tables
equivalence,
Our third
represented b

	p	q
1	T	T
2	T	I
3	T	F
4	I	T
5	I	I
6	I	F
7	F	T
8	F	I
9	F	F

As an i
8 express
is false a
then the c
disjunctio
implies q"
indetermin
true. As
suppose "
be false),
in 17A.D.
statement,
17"; we
from our

Three-Valued Logic

A two-valued logic system has its merits, simplicity foremost among them. It is interesting, nonetheless, to study the effect of including a third value for propositions whose truth or falsity is indeterminate (an "I don't know..." value). Often we find that the proof of a complicated theorem depends upon another complex proposition, a proposition whose truth is not readily ascertained. How may we determine the effect of this indeterminate proposition upon our hypothesis? First it is necessary to assign this indeterminate a logical value, and develop certain rules to handle this third value. Let us suppose we wished to prove that every odd number may be expressed as the sum of three primes. The proof of this theorem depends, obviously, upon the following proposition: every even number can be expressed as the sum of two primes. This proposition, however, has never been proved either true or false. Adapting a three-valued logic system to this problem, we should first assign as indeterminate the even number proposition. Then we could see what effect it has on our odd number hypothesis.

We set up a three-valued logic system with tables for conjunction, disjunction, equivalence, negation, and implication. Our third value is indeterminate, here represented by "I":

	p	q	$p \wedge q$	$p \vee q$	$p \leftrightarrow q$	$p \rightarrow q$	$\sim p$
1	T	T	T	T	T	T	F
2	T	I	I	T	I	I	F
3	T	F	F	T	F	F	F
4	I	T	I	T	I	T	I
5	I	I	I	I	I	I	I
6	I	F	F	I	I	I	I
7	F	T	F	T	F	T	T
8	F	I	F	I	I	T	T
9	F	F	F	F	T	T	T

As an illustration of this system, line 8 expresses the following: if statement p is false and statement q is indeterminate, then the conjunction (p and q) is false, the disjunction (p or q) is indeterminate, "p implies q" is true, "p is equivalent to q" is indeterminate, and "the negation of p" is true. As an example of these principles, suppose "p" is the statement (known to be false), "Rome was at war with Greece in 17A.D.," and "q" is the indeterminate statement, "A Roman was killed June 23, 17"; we then obtain the following result from our table: "Rome was at war with Greece AND a Roman was killed on June 23, 17" is false even though part of it is indeterminate; "IF Rome was at war with Greece THEN a Roman was killed on June 23, 17" is true whether or not the conclusion is ambiguous.

Although it is quite different from two-valued logic, there are many tautologies true in three-valued logic that are identical to or extensions of two-valued tautologies. The reader may easily verify the following by means of truth tables:

(1) $\sim(p \wedge q) \leftrightarrow [(\sim p) \vee (\sim q)]$
(2) $\sim(p \vee q) \leftrightarrow [(\sim p) \wedge (\sim q)]$
(3) $\sim(p \leftrightarrow q) \leftrightarrow [p \leftrightarrow (\sim q)]$
(4) $(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$.

It is interesting that the law of the syllogism is not valid in three-valued logic.

Concerning methods of proof, the rule of detachment and the rule of substitution for propositions are both valid.

Using these principles, and theorems one through four, we can work out a simple problem.

Given: p is false
q is indeterminate
$\sim(p \wedge q) \rightarrow r$
$r \leftrightarrow (\sim s)$.
Prove: s is false.

1. $\sim(p \wedge q) \leftrightarrow [(\sim p) \vee (\sim q)]$, but, since p is false, $\sim p$ is true. Therefore, from lines 1-3, $[(\sim p) \vee (\sim q)]$ is true. Substituting in the given, r is true by the rule of detachment.
2. Since r is true, $\sim s$ is also true, from line 1.
3. Therefore, s is false, from 7-9.

Problems

1. Given: p is indeterminate
q is true
Prove: $p \rightarrow q$
2. Given: $p \leftrightarrow q$ is indeterminate
p is true
$r \rightarrow q$
Prove: r is false

RALPH MILLER

---

1. Given two concentric circles of radii 17 and 8. A chord of the larger is tangent to the smaller circle. Find the length of the chord.

TRUE-FALSE LOGIC

(continued)

each given implication. They are known as the converse, inverse, and contrapositive. The formation of the converse consists of reversing the terms of an implication. The converse of $p \to q$ is $q \to p$. The inverse is formed by negating both sides of an implication. The inverse is $(\sim p) \to (\sim q)$. The contrapositive of $p \to q$ is $(\sim q) \to (\sim p)$; it is always equivalent to the original proposition, as can be easily verified from the following table:

p	q	$p \to q$	$q \to p$	$(\sim p) \to (\sim q)$	$(\sim q) \to (\sim p)$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Notice that the converse and inverse, while not always equivalent to the original proposition, are logically equivalent to each other, since they are contrapositives. For example, let $p \to q$ be "If a triangle be equilateral, then it is isosceles." This is true; we can now form the following: the converse is "If a triangle is isosceles, it is equilateral." The inverse is, "If a triangle is not equilateral, it is not isosceles." and the contrapositive is, "If a triangle is not isosceles, it is not equilateral." As is obvious, the inverse and converse are false, while the contrapositive is true.

The ultimate purpose of the logic system is application to specific situations, according to set rules. The first is the rule of detachment, which states that, if p is true and if $p \to q$ is true, then q is true. Referring to our first truth table, if p is true, we are in lines 1 or 2. If $p \to q$ is true, we are in 1, 3, or 4. The only line common to both is line 1, in which q is true, thus proving our rule. The rule of substitution for propositions states that any proposition may be substituted for its logical equivalent without changing the value of the proposition into which the substitution is made. For example, the symbolic expression of the rule of detachment is $[p \wedge (p \to q)] \to q$. Since we know that $(p \to q) \leftrightarrow [(\sim p) \vee q]$, we may substitute in the rule of detachment. $\{p \wedge [(\sim p) \vee q]\} \to q$. Simplifying, using a distributive law which we shall not state or prove (see Allendoerfer and Oakley, *Principles of Mathematics*): $\{[p \wedge (\sim p)] \vee (p \wedge q)\} \to q$. Since $p \wedge (\sim p)$ is obviously

continued on page 15

What Is The Calculus?

The calculus is the "beginning of higher mathematics," and, as such, presents a challenge to the serious student of mathematics. The calculus marks a transition between the set values of elementary math and the abstract variables of more advanced work.

Of all branches of mathematics, the calculus is the most useful. Engineering, astronomy, statistics, physics, and advanced mathematics are dependent upon it. The calculus is usually divided into two sub-branches, differential calculus and integral calculus. The differential calculus is concerned with the rate of change of function. For instance, if the distance traveled by a moving car is expressed in terms of the time which has elapsed, the velocity of the car could be found with the use of differential calculus. Illustrated graphically, this is equivalent to finding the slope of a curve at any point.

The integral calculus is used to find the value of a function when (1) its rate of change is known, and (2) its value at one instant (point) is given. The integral calculus enables us to find the areas, volumes, and lengths of figures all of which can be analysed by geometric methods only with great difficulty. Complex problems, arising in physics and vector mechanics, can be solved only with the use of the integral calculus. A graphic illustration of its use is in finding the area between a given curve and a given line.

The integral and differential calculi were thought to be independent of each other until Newton and Leibniz, working separately, discovered that these branches are really closely related. This discovery was revolutionary in that it marked the consolidation of the calculus into a powerful mathematical tool.

The discovery of this relationship is expressed in the fundamental theorem of the calculus. The introduction of this theorem, and Leibniz's development of a convenient system of notation for the calculus, were two of the most important mathematical achievements of the Renaissance, for the usefulness of the calculus was greatly increased. Thus, the way was paved for the development of mathematical physics and higher mathematics.

ERIC GANS

TRUE-F
(continued)

false, we h
$(p \wedge q) \to q$, wl

Let us so
Giv

Pro
Since $q \leftrightarrow r$,
by the rule
r is true.

1. Prove: {
2. Given: ~
Prove: r

Proble

2. Find
duct:

x =

3. Solve

10 log

4. The
triang
the h

5. Cows
cost
actly
many

6. Find
will
N^2 (I

7. A ma
of wa
quart
must
or fr
compl

8. Trian
and
with
(AC).

?

.ing of higher
, presents a
ent of mathe-
a transition
mentary math
nore advanced

.ematics, the
Engineering,
.cs, and ad-
ident upon it.
ded into two
calculus and
erential cal-
ate of change
the distance
expressed in
elapsed, the
e found with
s. Illustrated
nt to finding
nt.

ed to find the
.) its rate of
value at one
integral cal-
eas, volumes,

ch can be ana-
ds only with
roblems, aris-
echanics, can
of the integral
ion of its use
ween a given

ential calculi
ndent of each
ibniz, working
hese branches
This discovery
it marked the
into a power-

elationship is
al theorem of
ction of this
elopment of a
on for the cal-
mportant math-
Renaissance,
calculus was
the way was
f mathematical
ics.

ERIC GANS

TRUE-FALSE LOGIC
(continued)

false, we have:
$(p \wedge q) \rightarrow q$, which is easily shown true.

Let us solve the following problem:

Given: p is true (1)
$p \rightarrow q$ (2)
$q \leftrightarrow r$ (3)

Prove: r is true.

Since $q \leftrightarrow r$, we substitute r for q in (2). Then by the rule of detachment, from (1) and (2), r is true.

Problems

1. Prove: $\{\sim[(\sim p) \vee q] \rightarrow [q \vee (\sim r)]\} \leftrightarrow \{q \vee [\sim(p \wedge r)]\}$
2. Given: $\sim p \rightarrow q$; q is false; $p \rightarrow \sim r$.
   Prove: r is false.

ERIC GANS

Problems

2. Find the value of the infinite product:

$$x = \sqrt[2]{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} \cdot \cdot \cdot$$

3. Solve for all values of x:

$$10^{\log_{10} 9} = 8x + 5$$

4. The medians to the legs of a right triangle are $2\sqrt{73}$ and $4\sqrt{13}$. Find the hypoteneuse of the triangle.

5. Cows cost $170 apiece, and chickens cost $70 each. If a man spends exactly $1930, how many cows and how many chickens does he buy?

6. Find x if it is the largest number that will always divide evenly into: $N^2(N^2 - 1)$, when N is an integer.

7. A man wants to put exactly 39 quarts of water into a tank. He has an 11-quart and a 19-quart jar. How often must he pour from one jar to the other or from either into the tank to accomplish his task?

8. Triangle ABC has sides AB=13, BC=6, and AC=10. Triangle DEF is formed with DE=2(AB), EF=BC, and DF=2(AC). *Find the area of triangle DEF.*

Problems
(continued)

9. Three cars each go on a 48-mile trip. The first car goes twice as fast as the second. The third car needs one hour less than the total traveling time of the first two. If the third car goes ten miles per hour slower than the first, how many hours does each car take for the trip?

10. Each of four men can do a job in one hour, working alone. The first man works 3 times as long as the second, and together they do one-third of the entire job. The third man spends as much time more than the first as the fourth spends less than 5 times the second. If the job is finished, how many minutes does each man work?

11. Given triangle ABC, whose sides are AB, BC, and CA, in the ratio 29:20:21. A new triangle DEF is formed, such that side EF=$\frac{1}{4}$(BC), side FD=$\frac{1}{7}$(CA), and side DE=8. If the area of triangle ABC is 840, find the area of triangle DEF.

12. A plumber has a roll of tape 110 yards long and one inch wide. He wants to wrap a piece of pipe 7 inches in diameter with the tape. Find the maximum number of feet of pipe that can be wrapped with the tape (use $\pi = \frac{22}{7}$.

13. Two bicycles, which are 30 miles apart along a straight road, start traveling toward each other; one moves at 5 miles per hour and the other at 10. A fly on the front fender of one starts at the same instant as the bicycles and travels at 20 m.p.h. If it flies continuously from one bicycle to the other, how many miles does it go before it is crushed when the bicycles come together?

All of the answers to the problems appear in the following numbers; some answers are repeated. The answers appear here in scrambled order.

0, 5, 9, 15, 20, 30, 40, ½, 8, 6, 13, 12, 24, 16, ±2, ±3, 10, $\frac{9}{2}$, $\frac{12}{11}$