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AN EXTENSION OF A THEOREM OF HAMADA ON THE CAUCHY PROBLEM WITH SINGULAR DATA[1]

BY LESLIE LAMPORT

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Introduction. Hamada [**1**] proved the following result about the propagation of singularities in the Cauchy problem for an analytic linear partial differential operator. Assume that the initial data are analytic at the point $\mathbf{0}$ except for singularities along a submanifold T of the initial surface containing $\mathbf{0}$. Let $K^{(1)}, \cdots, K^{(m)}$ be the characteristic surfaces of the operator emanating from T. Under the assumption that the $K^{(i)}$ have multiplicity one, he showed that the solution of the Cauchy problem is analytic at $\mathbf{0}$ except for logarithmic singularities along the $K^{(i)}$. We extend his result to the case where the $K^{(i)}$ have constant multiplicity.

1. **Definitions and theorem.** Let $\boldsymbol{C}^{n+1}$ denote the set of $(n+1)$-tuples $\boldsymbol{x} = (x^0, \ldots, x^n)$ of complex numbers. Let S be an n-dimensional analytic submanifold of $\boldsymbol{C}^{n+1}$, and let T be an $(n-1)$-dimensional analytic submanifold of S. Since our results are local, we can assume $S = \{(0, x^1, \ldots, x^n) \in \boldsymbol{C}^{n+1}\}$ and $T = \{(0, 0, x^2, \ldots, x^n) \in \boldsymbol{C}^{n+1}\}$.

Let $D_i = \partial/\partial x^i$, $\boldsymbol{D} = (D_0, \ldots, D_n)$, and let $a : \boldsymbol{x} \to a(\boldsymbol{x}; \boldsymbol{D})$ be an analytic partial differential operator on a neighborhood of $\mathbf{0}$ in $\boldsymbol{C}^{n+1}$. Let $h(\boldsymbol{x}; \boldsymbol{D})$ be the principal part of $a(\boldsymbol{x}; \boldsymbol{D})$. We assume that S is not a characteristic surface of a at $\mathbf{0}$, so $h(\mathbf{0}; 1, 0, \ldots, 0) \neq 0$. Let $\boldsymbol{p} = (p_0, \ldots, p_n)$ be an $(n+1)$-tuple of formal variables, so $h(\boldsymbol{x}; \boldsymbol{p})$ is a homogeneous polynomial in $\boldsymbol{p}$ with analytic coefficients.

We say that the operator a has *constant multiplicity* at $\mathbf{0}$ in the direction of T if we can factor h as

$$h(\boldsymbol{x}; \boldsymbol{p}) = [h_1(\boldsymbol{x}; \boldsymbol{p})]^{k_1} \cdots [h_s(\boldsymbol{x}; \boldsymbol{p})]^{k_s}$$

for all $\boldsymbol{x}$ in a neighborhood of $\mathbf{0}$, where each $h_i(\boldsymbol{x}; \boldsymbol{p})$ is a polynomial in $\boldsymbol{p}$ of degree m_i with analytic coefficients, and the Σm_i roots of the polynomials $h_i(\mathbf{0}; \tau, 1, 0, \ldots, 0)$ in τ are all distinct. If $s = k_1 = 1$, then a is said to be of *multiplicity one* at $\mathbf{0}$ in the direction of T.

Assume now that a has constant multiplicity at $\mathbf{0}$ in the direction of T. It can be shown that we can find $m = \Sigma m_i$ analytic *characteristic functions* $\varphi^{(1)}, \ldots, \varphi^{(m)}$ of h defined in a neighborhood N of $\mathbf{0}$ satisfying:

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[1]The results described here are contained in the author's 1972 Ph.D. dissertation, written at Brandeis University under the supervision of Professor Richard Palais.

1. $h(\boldsymbol{x}; D\,\varphi^{(i)}(\boldsymbol{x})) = 0$ for all $\boldsymbol{x} \in N$.
2. $\varphi^{(i)}(0, x^1, \ldots, x^n) = x^1$ for all $(0, x^1, \ldots, x^n) \in N \cap S$.
3. For each $\boldsymbol{y} \in N \cap S$, the m numbers $D_0\,\varphi^{(i)}(\boldsymbol{y})$ are distinct.

Note that this implies that the numbers $D_0\,\varphi^{(i)}(\boldsymbol{y})$ are the distinct roots of the polynomials $h(\boldsymbol{y}; \tau, 1, 0, \ldots, 0)$ for each $\boldsymbol{y} \in N \cap S$. Let $K^{(i)} = \{\boldsymbol{x} : \varphi^{(i)}(\boldsymbol{x}) = 0\}$, so each $K^{(i)}$ is a characteristic surface of a.

Using these notations, we now state our result.

THEOREM. *Let a, N, S, T, $\varphi^{(i)}$ and $K^{(i)}$ be as above. Let v be an analytic function on N, and let w^j be an analytic function on $N \cap (S - T)$ for $j = 0, \ldots, r - 1$, where r is the degree of the operator a. Then there exists a neighborhood U of $\mathbf{0}$ such that the Cauchy problem*

(1) $a(\boldsymbol{x}; D)u(\boldsymbol{x}) = v(\boldsymbol{x}), \qquad (D_0)^j u(\boldsymbol{y}) = w^j(\boldsymbol{y}), \quad$ *for* $\boldsymbol{y} \in S, j = 0, \ldots, r - 1,$

has a solution u of the form

$$u(\boldsymbol{x}) = \sum_{i=1}^{m} F^{(i)}(\boldsymbol{x}) + G^{(i)}(\boldsymbol{x}) \log [\varphi^{(i)}(\boldsymbol{x})],$$

where each $F^{(i)}$ is analytic on $U - K^{(i)}$ and each $G^{(i)}$ is analytic on U.

Hamada proved this result when a has multiplicity one. In this case, if each w^j has at most a polar singularity along T, then each $F^{(i)}$ has at most a polar singularity along $K^{(i)}$. This is false in the general case, as is shown by the solution

$$u(t, y) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \frac{t^{2k+1}}{y^{k+1}}$$

of the two-dimensional Cauchy problem

$$\frac{\partial^2 u}{\partial t^2}(t, y) - \frac{\partial u}{\partial y}(t, y) = 0, \qquad u(0, y) = 0, \qquad \frac{\partial u}{\partial t}(0, y) = \frac{1}{y}.$$

2. **Method of proof.** The problem is easily reduced to solving the Cauchy problem (1) with each $w^j \equiv 0$ and v analytic on $N - K^{(1)}$. It can be shown that we may also assume that $h(\boldsymbol{x}; \boldsymbol{p}) = h_1(\boldsymbol{x}; \boldsymbol{p}) \cdots h_s(\boldsymbol{x}; \boldsymbol{p})$, where each h_i has multiplicity one in the direction of T and has $\varphi^{(1)}, \ldots, \varphi^{(m)}$ as characteristic functions (so $r = ms$).

Let the functions f_k be the ones defined by Hamada satisfying $df_k/dt = f_{k-1}$, for all integers k, and $f_0(t) = \log t$. The first step is to show that there exists a neighborhood V of $\mathbf{0}$ such that if v is of the form

$$v(\boldsymbol{x}) = \sum_{i=1}^{m} \sum_{k=0}^{\infty} v_k^{(i)}(\boldsymbol{x}) f_{k-l} [\varphi^{(i)}(\boldsymbol{x})], \tag{2}$$

with each $v_k^{(i)}$ analytic on V, then the Cauchy problem

$$h_i(x;D)u(x) = v(x), \qquad (D_0)^j u(y) = 0, \quad \text{for } y \in S, j = 0, \ldots, m-1,$$

has a formal series solution of the form

$$u(x) = \sum_{i=1}^{m} \sum_{k=0}^{\infty} u_k^{(i)}(x) f_{k-l+m-1}\,[\varphi^{(i)}(x)],$$

with each $u_k^{(i)}$ analytic on V. Moreover, bounds are obtained for the partial derivatives of the $u_k^{(i)}$ in terms of those of the $v_k^{(i)}$. This procedure is similar to the one used by Hamada.

Employing this result s times shows that with v given by (2), the Cauchy problem

$$h_1(x;D) \cdots h_s(x;D)u(x) = v(x), \ (D_0)^j u(y) = 0, \quad \text{for } y \in S, j = 0, \ldots, r-1,$$

has a formal solution

$$u(x) = \sum_{i=1}^{m} \sum_{k=0}^{\infty} u_k^{(i)}(x) f_{k-l+r-s}[\varphi^{(i)}(x)]$$

with the $u_k^{(i)}$ analytic on V. Again, bounds are obtained on the $u_k^{(i)}$.

Now we write $a(x;D) = h_1(x;D) \cdots h_s(x;D) + b(x;D)$, where the degree of b is less than r. Using the above results, we solve the sequence of Cauchy problems

$$h_1(x;D) \cdots h_s(x;D)_q u(x) = \begin{cases} v(x) & \text{if } q = 0, \\ -b(x;D)_{q-1}u(x) & \text{if } q > 0. \end{cases}$$

$$(D_0)^j{}_q u(y) = 0, \quad \text{for } y \in S, j = 0, \ldots, r-1,$$

to get

(3)
$$_q u(x) \doteq \sum_{i=1}^{m} \sum_{k=0}^{\infty} {}_q u_k^{(i)}(x) f_{k-l-q(s-1)}[\varphi^{(i)}(x)]$$

with each $_q u_k^{(i)}$ analytic on V. Then

(4)
$$u(x) = \sum_{q=0}^{\infty} {}_q u(x)$$

is easily seen to be a formal solution of (1) (with $w^j \equiv 0$).

Now assume $v(x) = v_l(x) f_{-l}[\varphi^{(1)}(x)]$, with v_l analytic on V, and let the corresponding solution (4) be $u_l(x) = \sum_{i=1}^{m} u_l^{(i)}(x)$. Using the bounds on the $_q u_k^{(i)}$, we can find a neighborhood W of $\mathbf{0}$ and demonstrate the absolute convergence of the sums (3) and (4) to prove that $u_l^{(i)}$ is analytic on $W - K^{(i)}$. Furthermore, we obtain a bound on $u_l^{(i)}$ in terms of a bound on v_l.

Finally, we can write $v(x) = \sum_{l=1}^{\infty} v_l(x) f_{-l}[\varphi^{(1)}(x)]$ (plus an analytic term which is handled by the Cauchy-Kowalewski theorem). It can be shown that there is a neighborhood U of $\mathbf{0}$ such that the sums $u^{(i)}(x) = \sum_{l=1}^{\infty} u_l^{(i)}(x)$

are absolutely convergent on $U - K^{(i)}$. It is then easily seen that the solution $u(x) = \sum_{i=1}^{m} u^{(i)}(x)$ has the desired form.

3. **Further generalizations.** It is evident from the proof that the theorem remains valid if v has a singularity along any of the hypersurfaces $K^{(i)}$. The theorem is also true if v has a singularity on any hypersurface K containing T which is not tangent to S or to any $K^{(i)}$ at $\mathbf{0}$.

By using different choices for the functions f_k, the result can be extended to the case where the w^j are p-valued analytic functions on $N \cap (S - T)$—i.e., multiple-valued functions finitely ramified about T—and v is a p-valued analytic function on $N - K^{(i)}$ or $N - K$. In this case, the $F^{(i)}$ become p-valued analytic functions on $U - K^{(i)}$. This result was also obtained by Wagschal [**2**] when a has multiplicity one.

References

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Massachusetts Computer Associates, Lakeside Office Park, Wakefield, Massachusetts 01880